Sequential Investment, Universal Portfolio Algos and Log-loss

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A market vector $\mathbf{x} = \{x_1, x_2, \ldots, x_m\}$ for $m$ assets is a vector of nonnegative real numbers representing price relatives for a given trading period.

$x_i \geq 0$ denotes the ratio of closing to opening price of the $i$th asset for that period.

An initial wealth invested in $m$ assets according to the fractions $Q_1, Q_2, \ldots, Q_m$ multiplies by a factor of $\sum_{i=1}^{m} x_i Q_i$ at the end of the period.

Market behavior during $n$ trading periods is represented by a sequence of market vectors $\mathbf{x}^n = (x_1, x_2, \ldots, x_n)$. 
The probability simplex in $\mathbb{R}^m$ is denoted by $\triangle_{m-1}$.

An investment strategy $Q$ for $n$ trading periods is a sequence $Q_1, \ldots, Q_n$ of vector valued functions $Q_t : \mathbb{R}_{+}^{t-1} \rightarrow \triangle_{m-1}$.

$i$th component $Q_{i,t}(\mathbf{x}^{t-1})$ of vector $Q_t(\mathbf{x}^{t-1})$ denotes the fraction of the current wealth invested in the $i$th asset at the beginning of the $t$th period on the basis of the past market behavior $\mathbf{x}^{t-1}$. 
Wealth Factor

\[ S_n(Q, x^n) = \prod_{t=1}^{n} \left( \sum_{i=1}^{m} x_{i,t} Q_{i,t}(x^{t-1}) \right) \]

denotes the \textit{wealth factor} of strategy \( Q \) after \( n \) trading periods.

- \( Q_t \) has nonnegative component summing to one expresses no short sales and no buying on margin.
Examples

- **Buy-and-Hold:**

\[
S_n(Q, x^n) = \sum_{j=1}^{m} Q_{j,1} \prod_{t=1}^{n} x_{j,t}
\]

\[
\leq \max_{j=1,...,m} \prod_{t=1}^{n} x_{j,t}
\]
### Examples

- **Constantly Rebalanced Portfolios:**
  - Parametrized by a probability vector
    \[ B = (B_1, B_2, \ldots, B_m) \in \triangle_{m-1} \]
  - \[ Q_t(x^{t-1}) = B \] regardless of \( t \) and \( x^{t-1} \)

\[
S_n(B, x^n) = \prod_{t=1}^{n} \left( \sum_{i=1}^{m} x_{i,t} B_i \right).
\]

- Example: \((1, \frac{1}{2}), (1, 2), (1, \frac{1}{2}), (1, 2), \ldots\)
  - Buy and Hold → No profit, No loss
  - CRP: \( B = (\frac{1}{2}, \frac{1}{2}) \rightarrow (\frac{9}{8})^{n/2} \), exponentially increasing wealth.
Minimax Wealth Ratio

Given a class $Q$ of investment strategies, the worst case logarithmic wealth ratio of a strategy $P$ is given by

$$W_n(P, Q) = \sup_{\bar{x}^n} \sup_{Q \in Q} \ln \frac{S_n(Q, \bar{x}^n)}{S_n(P, \bar{x}^n)}.$$

Minimax logarithmic wealth ratio is defined as:

$$W_n(Q) = \inf_P W_n(P, Q).$$

$W_n(P, Q) = o(n)$ means strategy $P$ achieves the same exponent of growth as the best reference strategy in class $Q$ for all market behaviors.
Prediction under log-loss and Investment

- Any investment strategy $Q$ can be used to define a forecaster that predicts elements $y_t \in \mathcal{Y}\{1, \ldots, m\}$ of a sequence $y^n \in \mathcal{Y}^n$ with probability vectors $\hat{p}_t \in \Delta_{m-1}$.

- **Kelly Market Vectors**: Market vectors $\bar{x}$ with a single component equal to 1 and all other components equal to zero.

- If $\bar{x}_1, \ldots, \bar{x}_n$ are Kelly market vectors, we denote the index of the only non-zero component of each vector $\bar{x}_t$ by $y_t$, we may define a forecaster $f$ by

$$f_t(y|y^{t-1}) = Q_{y,t}(\bar{x}^{t-1}).$$

- $f$ is induced by investment strategy $Q$. 
When $x^n$ is a sequence of Kelly vectors determined by the indices $y^n$, we write $S_n(Q, y^n)$ for $S_n(Q, x^n)$.

Note that $S_n(Q, y^n) = f_n(y^n)$, where $f$ is the forecaster induced by $Q$, where $f_n(y^n) = \prod_{t=1}^{n} f_t(y_t | y^{t-1})$, where $\sum_{y^n \in \mathcal{Y}^n} f_n(y^n) = 1$.

Conversely, given a $f_n(y^n)$, we may define

$$f_t(y_t | y^{t-1}) = \frac{f_t(y^t)}{f_{t-1}(y^{t-1})},$$

where $f_t(y^t) = \sum_{y^n_{t+1} \in \mathcal{Y}^{n-t}} f_n(y^n)$. 
Prediction under log-loss and Investment

- Log-loss: \( I(f_t, y_t) = -\ln f_t(y_t | y^{t-1}) \)
- Regret against a reference forecaster \( f \) is

\[
\hat{L}_n - L_{f,n} = \ln \frac{f_n(y^n)}{\hat{p}_n(y^n)} = \ln \frac{Q(y^n)}{P(y^n)},
\]

where \( Q \) and \( P \) are the investment strategies induced by \( f \) and \( \hat{p} \).
Lemma

Let $Q$ be a class of investment strategies, and let $\mathcal{F}$ denote the class of forecasters induced by the strategies in $Q$. Then, the minimax regret

$$V_n(\mathcal{F}) = \inf_{p^n} \sup_{y^n} \sup_{f \in \mathcal{F}} \ln \frac{f_n(y^n)}{p_n(y^n)}$$

satisfies $W_n(Q) \geq V_n(\mathcal{F})$. 
Proof.

Let $P$ be any investment strategy and let $p$ be it’s induced forecaster. Then

$$\sup_{\mathcal{X}^n} \sup_{Q \in \mathcal{Q}} \ln \frac{S_n(Q, \mathcal{X}^n)}{S_n(P, \mathcal{X}^n)} \geq \max_{y^n \in Y^n} \sup_{Q \in \mathcal{Q}} \ln \frac{S_n(Q, y^n)}{S_n(P, y^n)}$$

$$\quad = \max_{y^n \in Y^n} \sup_{f \in \mathcal{F}} \ln \frac{f_n(y^n)}{p_n(y^n)}$$

$$\quad = V_n(p, \mathcal{F}) \geq V_n(\mathcal{F}).$$
Given a prediction $p$, we define an investment strategy $P$ as follows:

$$P_{j,t}(x^{t-1}) = \frac{\sum_{y^{t-1} \in Y^{t-1}} p_t(j|y^{t-1})p_{t-1}(y^{t-1})(\prod_{s=1}^{t-1} x_{y_s,s})}{\sum_{y^{t-1} \in Y^{t-1}} p_{t-1}(y^{t-1})(\prod_{s=1}^{t-1} x_{y_s,s})}$$

- The obtained investment strategy induces $p$, and so we say $p$ and $P$ induce each other.
- $\prod_{s=1}^{t-1} x_{y_s,s}$ may be viewed as the return of the extremal investment strategy that, on each trading period $t$, invests everything on the $y_t$ th asset.
Theorem

Let $P$ be an investment strategy induced by a forecaster $p$, and let $Q$ be an arbitrary class of investment strategies. Then for any market sequence $x^n$,

$$\sup_{Q \in Q} \ln \frac{S_n(Q, x^n)}{S_n(P, x^n)} \leq \max_{y^n \in Y^n} \sup_{Q \in Q} \ln \frac{\prod_{t=1}^{n} Q_{y,t}(x^{t-1})}{p_n(y^n)}$$
Lemma

Let \( a_1, \ldots, a_n, b_1, \ldots, b_n \) be non-negative numbers. Then,

\[
\frac{\sum_{i=1}^{n} a_i}{\sum_{i=1}^{n} b_i} \leq \max_{j=1,\ldots,n} \frac{a_j}{b_j},
\]

where we define \( 0/0 = 0 \).
Lemma

The wealth factor achieved by an investment strategy $Q$ may be written as

$$S_n(Q, x^n) = \sum_{y^n \in \mathcal{Y}^n} \left( \prod_{t=1}^{n} x_{y,t} \right) \left( \prod_{t=1}^{n} Q_{y,t} (x^{t-1}) \right).$$

If the investment strategy $P$ is induced by a forecaster $p_n$, then

$$S_n(P, x^n) = \sum_{y^n \in \mathcal{Y}^n} \left( \prod_{t=1}^{n} x_{y,t} \right) p_n(y^n).$$
Proof.

\[ S_n(Q, \bar{x}^n) = \prod_{t=1}^{n} \left( \sum_{j=1}^{m} x_{j,t} Q_{j,t}(\bar{x}^{t-1}) \right) \]

\[ = \sum_{y^n \in \mathcal{Y}^n} \left( \prod_{t=1}^{n} x_{y_t,t} Q_{y_t,t}(\bar{x}^{t-1}) \right) \]

\[ = \sum_{y^n \in \mathcal{Y}^n} \left( \prod_{t=1}^{n} x_{y_t,t} \right) \left( \prod_{t=1}^{n} Q_{y_t,t}(\bar{x}^{t-1}) \right). \]
Proof.

\[ S_n(P, x^n) = \prod_{t=1}^{n} \left( \sum_{j=1}^{m} x_{j,t} P_{j,t}(x^{t-1}) \right) \]

\[ = \prod_{t=1}^{n} \frac{\sum_{j=1}^{m} \sum_{y^{t-1} \in \mathcal{Y}^{t-1}} p_t(y^{t-1}j) x_{j,t} \left( \prod_{s=1}^{t-1} x_{y_s,s} \right)}{\sum_{y^{t-1} \in \mathcal{Y}^{t-1}} p_t(y^{t-1}j) x_{j,t} \left( \prod_{s=1}^{t-1} x_{y_s,s} \right)} \]

\[ = \prod_{t=1}^{n} \frac{\sum_{y^t \in \mathcal{Y}^t} \left( \prod_{s=1}^{t} x_{y_s,s} \right) p_t(y^t)}{\sum_{y^{t-1} \in \mathcal{Y}^{t-1}} \left( \prod_{s=1}^{t-1} x_{y_s,s} \right) p_{t-1}(y^{t-1})} \]

\[ = \sum_{y^n \in \mathcal{Y}^n} \left( \prod_{t=1}^{n} x_{y_t,t} \right) p_n(y^n). \]
Proof.

Fix any market sequence $x^n$ and choose any reference strategy $Q' \in Q$. Denote by $S_n(y^n, x^n) = \prod_{t=1}^{n} x_{y_t,t}$, then

$$
\frac{S_n(Q', x^n)}{S_n(P, x^n)} = \frac{\sum_{y^n \in \gamma^n} S_n(y^n, x^n)(\prod_{t=1}^{n} Q'_{y_t,t}(x^{t-1}))}{\sum_{y^n \in \gamma^n} S_n(y^n, x^n)p_n(y^n)}
$$

$$
\leq \max_{y^n: S_n(y^n, x^n) > 0} \frac{S_n(y^n, x^n)(\prod_{t=1}^{n} Q'_{y_t,t}(x^{t-1}))}{S_n(y^n, x^n)p_n(y^n)}
$$

$$
= \max_{y^n \in \gamma^n} \frac{\prod_{t=1}^{n} Q'_{y_t,t}(x^{t-1})}{p_n(y^n)}
$$

$$
\leq \max_{y^n \in \gamma^n} \sup_{q \in Q} \frac{\prod_{t=1}^{n} Q_{y_t,t}(x^{t-1})}{p_n(y^n)}.
$$
Theorem

Let $Q$ be a class of static investment strategies, and let $\mathcal{F}$ denote the class of forecasters induced by strategies in $Q$. Then

$$W_n(Q) = V_n(\mathcal{F}).$$

Furthermore, the minimax optimal investment strategy is defined by

$$P_{j,t}(x^{t-1}) = \frac{\sum_{y^{t-1} \in Y^{t-1}} p^*_t(j | y^{t-1}) p^*_{t-1}(y^{t-1}) (\prod_{s=1}^{t-1} x_{y_s,s})}{\sum_{y^{t-1} \in Y^{t-1}} p^*_{t-1}(y^{t-1}) (\prod_{s=1}^{t-1} x_{y_s,s})}$$

where $p^*$ is the normalized maximum likelihood forecaster

$$p^*_n(y^n) = \frac{\sup_{Q \in Q} \prod_{t=1}^{n} Q_{y_t,t}}{\sum_{y^n \in Y^n} \sup_{Q \in Q} \prod_{t=1}^{n} Q_{y_t,t}}.$$
Definition

The normalized maximum likelihood forecaster is defined by the following:

\[ p_n^*(y^n) = \frac{\sup_{f \in \mathcal{F}} f_n(y^n)}{\sum_{x^n \in \mathcal{Y}^n} \sup_{f \in \mathcal{F}} f_n(y^n)} \]
Normalized Maximum Likelihood Forecaster

**Theorem**

For any class $\mathcal{F}$ of experts and integer $n > 0$, the normalized maximum likelihood forecaster $p^*$ is the unique forecaster such that

$$\sup_{y^n \in \mathcal{Y}^n} (\hat{L}(y^n) - \inf_{f \in \mathcal{F}} L_f(y^n)) = V_n(\mathcal{F}).$$

Moreover, $p^*$ is an equalizer that is, for all $y^n \in \mathcal{Y}^n$,

$$\ln \sup_{f \in \mathcal{F}} f_n(y^n) = \ln \sum_{x^n \in \mathcal{Y}^n} \sup_{f \in \mathcal{F}} f_n(x^n) = V_n(\mathcal{F}).$$
Proof.

The normalized maximum likelihood forecaster $p^*$ is minimax optimal for the class $\mathcal{F}$; that is,

$$\max_{y^n \in \mathcal{Y}^n} \ln \sup_{Q \in \mathcal{Q}} \prod_{t=1}^n Q_{y^t,t} \frac{p^*_n(y^n)}{p^*_n(y^n)} = V_n(\mathcal{F}).$$

Now, let $P^*$ be the investment strategy induced by minimax forecaster $p^*$ for $Q$. By theorem, we get

$$W_n(Q) \leq \sup_{x^n} \sup_{Q \in \mathcal{Q}} \ln \frac{S_n(Q, x^n)}{S_n(P^*, x^n)} \leq \max_{y^n \in \mathcal{Y}^n} \sup_{Q \in \mathcal{Q}} \ln \frac{\prod_{t=1}^n Q_{y^t,t}}{p^*_n(y^n)} = V_n(\mathcal{F}).$$
Constantly Rebalanced Portfolios

\[ W_n(Q) = \frac{m-1}{2} \ln n + \ln \frac{\Gamma(1/2)^m}{\Gamma(m/2)} + o(1) \]
Universal Portfolios

- We restrict our attention to class $Q$ of all constantly rebalanced portfolios.
- Each strategy $Q$ in this class is determined by a vector $\mathcal{B} = \{B_1, B_2, \ldots, B_m\} \in \triangle_{m-1}$.
- The *Universal Portfolio* strategy $P$ is given by

$$P_{j,t}(x^{t-1}) = \frac{\int_{\triangle_{m-1}} B_j S_{t-1}(B, x^{t-1}) \mu(B) dB}{\int_{\triangle_{m-1}} S_{t-1}(B, x^{t-1}) \mu(B) dB},$$

where $j = 1, 2, \ldots, m$, $t = 1, \ldots, n$, and $\mu$ is a density function on $\triangle_{m-1}$. 
Universal Portfolios

The wealth achieved by the universal portfolio is just the average of the wealths achieved by the individual strategies in the class.

\[
S_n(P, x^n) = \prod_{t=1}^{n} \sum_{j=1}^{m} P_{j,t}(x^{t-1})x_{j,t}
\]

\[
= \prod_{t=1}^{n} \frac{\int_{\Delta_{m-1}} \sum_{j=1}^{m} x_{j,t} B_j S_{t-1}(B, x^{t-1}) \mu(B) d\mathcal{B}}{\int_{\Delta_{m-1}} S_{t-1}(B, x^{t-1}) \mu(B) d\mathcal{B}}
\]

\[
= \prod_{t=1}^{n} \frac{\int_{\Delta_{m-1}} S_t(B, x^t) \mu(B) d\mathcal{B}}{\int_{\Delta_{m-1}} S_{t-1}(B, x^{t-1}) \mu(B) d\mathcal{B}}
\]

\[
= \int_{\Delta_{m-1}} S_n(B, x^n) \mu(B) d\mathcal{B}
\]
Universal Portfolios

It’s like a Buy-and-Hold on all Constantly Rebalanced Portfolios (CRP)

\[ S_n(P, x^n) = \int_{\Delta_{m-1}} S_n(B, x^n) \mu(B) dB \]

If it helps, think of it as: (Riemann sum approximation)

\[ S_n(P, x^n) = \sum_i Q_i S_n(B_i, x^n), \]

where, given the elements \( \Delta_i \) of a fine partition of the simplex \( \Delta_{m-1} \), we assume that \( B_i \in \Delta_{m-1} \) and \( Q_i = \int_{\Delta_i} \mu(B) dB \).
Theorem

If $\mu$ is the uniform density on the probability density simplex $\triangle_{m-1} \in \mathbb{R}^m$, then the wealth achieved by the universal portfolio satisfies

$$\sup_{\bar{x}^n} \sup_{B \in \triangle_{m-1}} \ln \frac{S_n(B, \bar{x}^n)}{S_n(P, \bar{x}^n)} \leq (m - 1) \ln(n + 1).$$

If the universal portfolio is defined using the Dirichlet$(1/2, \ldots, 1/2)$ density $\mu$, then

$$\sup_{\bar{x}^n} \sup_{B \in \triangle_{m-1}} \ln \frac{S_n(B, \bar{x}^n)}{S_n(P, \bar{x}^n)} \leq \frac{m - 1}{2} \ln n + \ln \frac{\Gamma(1/2)^m}{\Gamma(m/2)} + \frac{m - 1}{2} \ln 2 + o(1).$$
EG investment strategy

- Universal Portfolio involves integration over $m$-dimensional simplex.
- EG’s computational cost is linear in $m$.
- EG investment strategy invests at a time $t$ using the vector $\bar{P}_t = (P_{1,t}, \ldots, P_{m,t})$ where $\bar{P}_t = (1/m, \ldots, 1/m)$ and

$$P_{i,t} = \frac{P_{i,t-1} \exp(\eta(x_{i,t-1}/P_{t-1} \cdot x_{t-1}))}{\sum_{j=1}^{m} P_{j,t-1} \exp(\eta(x_{j,t-1}/P_{t-1} \cdot x_{t-1}))}$$

where $i = 1, 2, \ldots, m$ and $t = 2, 3, \ldots$. 
EG investment strategy

Special case of gradient-based forecaster:

\[ P_{i,t} = \frac{P_{i,t-1} \exp(\eta \nabla \ell_{t-1}(P_{t-1})_i)}{\sum_{j=1}^{m} P_{j,t-1} \exp(\eta \nabla \ell_{t-1}(P_{t-1})_j)} \]

when the loss function is set as \( \ell_{t-1}(P_{t-1}) = -\ln P_{t-1} \cdot x_{t-1} \).
Theorem

Assume that the price relatives $x_{i,t}$ all fall between two positive constants $c < C$. Then the worst-case logarithmic wealth ratio of the EG investment strategy with $\eta = (c/C)\sqrt{(8 \ln m)/n}$ is bounded by

$$\frac{\ln m}{\eta} + \frac{m \eta}{8} \frac{C^2}{c^2} = \frac{C}{c} \sqrt{\frac{n}{2}} \ln m.$$
Main Idea: Portfolios that are “near” each other perform similarly, and there is a large fraction of portfolios “near” the optimal one.

Suppose in hindsight $B^*$ is the optimal CRP. Let $B = (1 - \alpha)B^* + \alpha z$, for some $z \in \triangle_{m-1}$.(Meaning, $B$ is close to $B^*$).

For a single period

$$\text{gain of } CRP_B \geq (1 - \alpha)(\text{gain of } CRP_{B^*}).$$

Over $n$ periods,

$$\text{wealth of } CRP_B \geq (1 - \alpha)^n(\text{wealth of } CRP_{B^*}).$$
Simple proof without transaction costs

\[
\frac{\text{wealth of UNIVERSAL}}{\text{wealth of best CRP}} \geq E_{B \in \triangle_{m-1}}[(1 - \alpha)^n] \\
= \int_0^1 \text{Prob}_{B \in \triangle_{m-1}}[(1 - \alpha)^n \geq x] \, dx \\
= \int_0^1 (1 - x^{1/n})^{m-1} \, dx \\
= n \int_0^1 y^{n-1} (1 - y)^{m-1} \, dy \\
= \ldots \\
= n \left( \frac{(m - 1)! (n - 1)!}{n + m - 2} \right)! \\
= \frac{1}{\binom{n+m-1}{m-1}}
\]
The following assumptions are made:

- The costs paid changing from distribution $B_1$ to $B_3$ is no more than the costs paid changing from $B_1$ to $B_2$ and then from $B_2$ to $B_3$.

- The cost, per dollars, of changing from a distribution $B$ to a distribution $(1 - \alpha)B_1 + \alpha B'$ is no more than $\alpha c$, because at most an $\alpha$ fraction of the money is being moved.

- An investment strategy $I$ which invests an initial fraction $\alpha$ of its money according to investment strategy $I_1$ and an initial $1 - \alpha$ of its money according to $I_2$, will achieve at least $\alpha$ times the wealth of $I_1$ plus $1 - \alpha$ times the wealth of $I_2$. 
Theorem

In the presence of commission $0 \leq c \leq 1$, 

$$\frac{\text{wealth of UNIVERSAL}_c}{\text{wealth of best CRP}} \geq \left(\frac{(1 + c)n + m - 1}{m - 1}\right)^{-1} \geq \frac{1}{((1 + c)n + 1)^{m-1}}.$$
Result with Commission

Proof.

Based on the properties we assumed, if \( B_j \geq (1 - \alpha)B_j^* \), then

\[
\frac{\text{single-period profit of CRP}_B}{\text{single-period profit of CRP}_{B^*}} \geq (1 - \alpha)(1 - c\alpha).
\]

Over \( n \) periods, this gives

\[
\text{wealth of CRP}_B \geq (1 - \alpha)^{(1+c)n}(\text{wealth of CRP}_{B^*}).
\]

The previous proof can be applied and we can replace \( n \) by \( (1 + c)n \) in the final guarantee.