Blackwell Approachability and Forcing Halfspaces
CSE 254: Online Learning

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Basic Results

Classic minimax theorem for two-player zero-sum games:

**Theorem (von Neumann, 1947)**

*If the players have discrete strategy spaces \([n], [m]\) and the game has payoff function \(u : [n] \times [m] \mapsto \mathbb{R}\),

\[
\max_{p \in \Delta_n} \min_{q \in \Delta_m} \sum_{i \in [n], j \in [m]} p_i q_j u(i, j) = \min_{q \in \Delta_m} \max_{p \in \Delta_n} \sum_{i \in [n], j \in [m]} p_i q_j u(i, j)
\]
Basic Results

"Optimization" variant

Theorem (Sion, 1958)

If the players have convex compact strategy spaces $\mathcal{X}, \mathcal{Y}$ and the game has loss function $f(x, y) : \mathcal{X} \times \mathcal{Y} \rightarrow \mathbb{R}$, convex in $x$ and concave in $y$,

$$\inf_{x \in \mathcal{X}} \sup_{y \in \mathcal{Y}} f(x, y) = \sup_{y \in \mathcal{Y}} \inf_{x \in \mathcal{X}} f(x, y)$$

Analogues for other games, nothing as powerful

Rich equilibrium structure impossible with more players

But can we go beyond scalar payoff functions?
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Preliminaries

- **Two-player zero-sum game**
  - Player $X$ plays against nameless adversary $Y$ (Nature)
  - $X$ plays $x \in X$, $Y$ plays $y \in Y$
  - $X$ loses $u(x, y)$, $Y$ wins $u(x, y)$

- **Minimax value** $V = \min_{x \in X} \max_{y \in Y} u(x, y)$ when $U$ is scalar

- **Vector-valued games**
  - Natural to model utility of mutually independent factors
  - What can we say when $u$ is vector-valued? Minimax impossible
Explicit Quantification

- Minimax (strong) duality is the conjunction of two statements involving value $V$:
  1. $\min_{x \in X} \max_{y \in Y} u(x, y) \leq V \iff \exists x \in X : \forall y \in Y : u(x, y) \leq V$
  2. $\max_{y \in Y} \min_{x \in X} u(x, y) \geq V \iff \exists y \in Y : \forall x \in X : u(x, y) \geq V$

and weak duality $\max_{y \in Y} \min_{x \in X} u(x, y) \leq \min_{x \in X} \max_{y \in Y} u(x, y)$.

- Each player can force the other into playing in a way that guarantees the payoff in a half-line.
- In this worst-case scenario, the only meaningful control is a uniform guarantee over adversary strategies.
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Setup

- What happens when the payoff is vector-valued?
- What payoffs can $X$ force the adversary into settling for?
- Can $X$ force payoffs in some target set? ¹

¹ (Not possible in a one-shot game even with convexity.)
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\(^1\) (Not possible in a one-shot game even with convexity.)
Blackwell’s Game

- A two-player zero-sum repeated game with vector-valued payoff $u(x, y)$
- On iteration $t$, $X$ plays $x_t$ first, then $Y$ plays $y_t$
- **Goal of player** $X$: “Approach" target set $S$ regardless of $Y$’s actions
- **Assumptions**
  - Any projection $u_\theta(x, y) = \langle \theta, u(x, y) \rangle$ for any vector $\theta$ satisfies minimax conditions (e.g. if $u$ is bilinear)
  - $S, X, Y$ are convex, compact
  - These are unnecessary in many cases
Definition: Approachability

Consider a set $S$. Define $S$ to be approachable if there exists a possibly adaptive strategy $x_1, x_2, x_3, \cdots \in X$ such that for any sequence $y_1, y_2, \cdots \in Y$,

$$\lim_{T \to \infty} d\left(\frac{1}{T} \sum_{t=1}^{T} u(x_t, y_t), S\right) = 0$$

where $d$ is the distance in Euclidean norm. In other words, if $\bar{u}_T = \frac{1}{T} \sum_{t=1}^{T} u(x_t, y_t)$,

$$\lim_{T \to \infty} \inf_{z \in S} \| \bar{u}_T - z \| = 0$$

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$^2$For simplicity, throughout only consider subsets of $\mathbb{R}^d$ for finite $d$. 
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A set $S$ is:

- **Satisfiable** (by $X$) if $\exists x \in X : \forall y \in Y : u(x, y) \in S$
  (player can force $S$ playing first)

- **Response-satisfiable** (by $X$) if $\forall y \in Y : \exists x \in X : u(x, y) \in S$
  (player can force $S$ playing second)

- Satisfiability $\implies$ response-satisfiability
- When does response-satisfiability $\implies$ satisfiability?
- Other relations hold ($S$ is satisfiable by $X$ $\iff$ $S^c$ is response-satisfiable by $Y$)
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Some Related Notions

A set $S$ is:

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Satisfiability for Halfspaces

- Minimax theorem: \((-\infty, c]\) is approachable \iff c \geq V
- Consider any halfspace \(H = \{s : \langle \theta, s \rangle \leq c\}\)
- This induces scalar game with payoff \(u_\theta(x, y) = \langle \theta, u(x, y) \rangle\)
- \(H\) is approachable
  \iff \((-\infty, c]\) is approachable in scalar game
  \iff c \geq \min_{x \in \mathcal{X}} \max_{y \in \mathcal{Y}} u_\theta(x, y)
  \iff \exists x \in \mathcal{X} : \forall y \in \mathcal{Y} : u(x, y) \in H
  \iff H\) is satisfiable
Minimax theorem: $(-\infty, c]$ is approachable $\iff c \geq V$

Consider any halfspace $H = \{s : \langle \theta, s \rangle \leq c \}$

This induces scalar game with payoff $u_\theta(x, y) = \langle \theta, u(x, y) \rangle$

$H$ is approachable

$\iff (-\infty, c]$ is approachable in scalar game

$\iff c \geq \min_{x \in X} \max_{y \in Y} u_\theta(x, y)$

$\iff \exists x \in X : \forall y \in Y : u(x, y) \in H$

$\iff H$ is satisfiable
Response-Satisfiability $\iff$ Halfspace-Satisfiability

Theorem

\( S \) is response-satisfiable $\iff$ every halfspace \( H \supseteq S \) is satisfiable.

Proof.

( $\Rightarrow$ )

Take any halfspace \( H_0 = \{ s : \langle \theta_0, s \rangle \leq c_0 \} \supseteq S \).

Now \( S \) is response-satisfiable $\Rightarrow \forall y : \exists x_y : u(x_y, y) \in S$ $\Rightarrow u(x_y, y) \in H_0$ $\Rightarrow u_{\theta_0}(x_y, y) \leq c_0$. Thus

\[ c_0 \geq \max_{y \in Y} u_{\theta_0}(x_y, y) \geq \max_{y \in Y} \min_{x \in X} u_{\theta_0}(x, y) = \min_{x \in X} \max_{y \in Y} u_{\theta_0}(x, y). \]

If \( x^* \) is the minimizer here, we have

\[ \forall y \in Y : c_0 \geq u_{\theta_0}(x^*, y) \Rightarrow u(x^*, y) \in H_0. \]
**Theorem**

\[ S \text{ is response-satisfiable} \iff \text{every halfspace } H \supseteq S \text{ is satisfiable.} \]

**Proof.**

\((\implies)\)

Take any halfspace \( H_0 = \{ s : \langle \theta_0, s \rangle \leq c_0 \} \supseteq S. \)

Now \( S \) is response-satisfiable \( \implies \forall y : \exists x_y : u(x_y, y) \in S \implies u(x_y, y) \in H_0 \implies u_{\theta_0}(x_y, y) \leq c_0. \) Thus

\[ c_0 \geq \max_{y \in Y} u_{\theta_0}(x_y, y) \geq \max_{y \in Y} \min_{x \in X} u_{\theta_0}(x, y) = \min_{x \in X} \max_{y \in Y} u_{\theta_0}(x, y). \]

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Proof.

\[(\Leftarrow)\]

\(S\) not response-satisfiable \iff \exists y_0 \in \mathcal{Y} : \forall x : u(x, y_0) \notin S.\)

The set \(U = \{ \forall x \in \mathcal{X} : u(x, y_0) \} \) is convex, but \(S \cap U = \emptyset\) by assumption. So there is a hyperplane \(H\) separating \(S\) and \(U\), defining a halfspace \(H \supseteq S\). We have for all \(x\) that \(u(x, y_0) \notin S \implies u(x, y_0) \notin H\), so \(H\) is not satisfiable.
Halfspace-Satisfiability $\iff$ Approachability

**Theorem**

*Every halfspace $H \supseteq S$ is satisfiable $\iff S$ is approachable.*

**Proof.**

$(\implies)$ Constructive; the algorithm that approaches $S$ relies on a halfspace oracle $O(H)$ for any $H \supseteq S$, with $O(H) = \{ x \in \mathcal{X} : \forall y \in \mathcal{Y} : u(x, y) \in H \}$.

$(\impliedby)$ $\exists H \supseteq S$ not satisfiable $\implies \exists H \supseteq S$ not approachable

$\implies S$ is not approachable
Every halfspace $H \supseteq S$ is satisfiable $\iff$ $S$ is approachable.

Proof.

( $\implies$ ) Constructive; the algorithm that approaches $S$ relies on a halfspace oracle $O(H)$ for any $H \supseteq S$, with

$$O(H) = \{x \in \mathcal{X} : \forall y \in \mathcal{Y} : u(x, y) \in H\}.$$

( $\impliedby$ ) $\exists H \supseteq S$ not satisfiable $\implies \exists H \supseteq S$ not approachable $\implies S$ is not approachable \hfill $\Box$
The following are equivalent characterizations of $S$:

1. Response-satisfiable
2. Halfspace-satisfiable
3. Approachable

The first is often used to derive the third.
Outline

1. Two-Player Zero-Sum Games
2. Blackwell Approachability
   - Approachability Basics
   - Related Notions
   - Blackwell’s Algorithm
3. Potential-Based Approachability and Algorithms
   - Potential-Based Approachability
   - Potential-Based Prediction Algorithms
   - Connections to Drifting Games and Online Learning
4. No-Regret Algorithms and Approachability
5. Summary
An Approachability Algorithm

- Assume oracle $O(H)$ for any $H \supseteq S$, with
  $O(H) = \{ x \in X : \forall y \in Y : u(x, y) \in H \}$

- Write $A_T = \frac{1}{T} \sum_{t=1}^{T} u(x_t, y_t)$, and the projection
  $\pi_S(A_t) = \arg \min_{v \in S} \| A_t - v \|

- Algorithm: On iteration $t$, if $A_{t-1} \notin S$, play $O(H_{t-1})$, where
  $H_{t-1} = \{ x : \forall y \in Y : \langle \frac{A_{t-1} - \pi_S(A_{t-1})}{\|A_{t-1} - \pi_S(A_{t-1})\|}, u(x, y) - \pi_S(A_{t-1}) \rangle \leq 0 \}$

- Assumptions: $\| u(x, y) \| \leq 1 \forall x, y$; $S$ is contained in the unit ball also
An Approachability Algorithm
Proof of Approachability (Algorithm)

\[ \|A_t - \pi_S(A_t)\|^2 \leq \|A_t - \pi_S(A_{t-1})\|^2 \]
\[ = \|A_t - A_{t-1}\|^2 + \|A_{t-1} - \pi_S(A_{t-1})\|^2 + 2 \langle A_{t-1} - \pi_S(A_{t-1}), A_t - A_{t-1} \rangle \]
\[ \leq \|A_t - A_{t-1}\|^2 + \|A_{t-1} - \pi_S(A_{t-1})\|^2 + 2 \langle A_{t-1} - \pi_S(A_{t-1}), A_t - A_{t-1} \rangle \]

Now \(A_t - A_{t-1} = \frac{u(x_t, y_t) - A_{t-1}}{t} = \frac{1}{t} ((u(x_t, y_t) - \pi_S(A_{t-1})) - (A_{t-1} - \pi_S(A_{t-1}))),\) so

\[ \langle A_{t-1} - \pi_S(A_{t-1}), A_t - A_{t-1} \rangle = \frac{1}{t} \langle A_{t-1} - \pi_S(A_{t-1}), u(x_t, y_t) - \pi_S(A_{t-1}) \rangle \]
\[ - \frac{1}{t} \langle A_{t-1} - \pi_S(A_{t-1}), A_{t-1} - \pi_S(A_{t-1}) \rangle \]
\[ \leq -\frac{1}{t} \|A_{t-1} - \pi_S(A_{t-1})\|^2 \]

Therefore \(\|A_t - \pi_S(A_t)\|^2 \leq \left(1 - \frac{2}{t}\right) \|A_{t-1} - \pi_S(A_{t-1})\|^2 + \frac{4}{t^2} \]
\[ \implies \|A_t - \pi_S(A_t)\|^2 \leq O\left(\frac{1}{t}\right). \]
Two-Player Zero-Sum Games
Blackwell Approachability
Potential-Based Approachability and Algorithms
No-Regret Algorithms and Approachability
Summary

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\[ \Rightarrow \|A_t - \pi_S(A_t)\|^2 \leq O \left( \frac{1}{t} \right) . \]
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   - Approachability Basics
   - Related Notions
   - Blackwell’s Algorithm
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   - Potential-Based Prediction Algorithms
   - Connections to Drifting Games and Online Learning
4. No-Regret Algorithms and Approachability
5. Summary
Generalizing Blackwell’s Strategy

- Keep track of **potential function** $\Phi(s)$ that measures distance to set $S$ ($\Phi(s) = 0 \ \forall s \in S$)
- Want to minimize $\Phi(R_t)$ whenever possible
- Idea: Force halfspace in the direction of $\nabla \Phi(A_{t-1})$, but translated to intersect $\pi_S(A_{t-1})$
- Blackwell strategy: $\Phi(x) = \inf_{y \in S} \|x - y\|^2$
- Loss bound $\Phi(A_t) \in O(\ln t / t)$
Generalizing Blackwell’s Strategy

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Potential-Based Approachability

The diagram illustrates the concept of potential-based approachability in a game context. The set \( \{ x : \Phi(x) = \text{const.} \} \) represents the set of states where the potential function \( \Phi \) is constant. The function \( \ell(p_t, J_t) \) likely denotes a loss function or a metric of some kind. The point \( A_{t-1} \) and the set \( S \) are key elements in defining the approachability of a strategy. The inequality \( \{ u : a_{t-1} \cdot u = c_{t-1} \} \) might represent constraints on the utility vector \( u \) for a given action \( a_{t-1} \) with a specific weight \( c_{t-1} \). The diagram suggests a dynamic interaction between these elements in the context of potential-based approachability.
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Potential-Based Prediction with Experts

- Prediction with expert advice
  - On iteration $t$, $N$ experts each predict in decision space $\mathcal{Z}$
  - Algorithm predicts $z_{A,t} \in \mathcal{Z}$, Nature reveals outcome $y_t$
  - Expert $i$ incurs loss $l_{i,t}$, algorithm incurs $l_{A,t}$
  - Instantaneous regret $r_{i,t} = l_{A,t} - l_{i,t}$ to expert $i$

- As a game with losses in $\mathbb{R}^N$, one expert per coordinate
  - $r_t \in \mathbb{R}^N$ is vector with components $r_{i,t}$; $R_t = \sum_{i=1}^{t} r_i$
  - Game loss at time $t$ is $u_t = r_t$

- Solved with a potential $\Phi(u) = \psi \left( \sum_{i=1}^{N} \phi(u_i) \right)$
  - $\phi$ nonnegative, increasing, twice-diff.
  - $\psi$ concave, nonnegative, strictly increasing, twice-diff.
  - Relaxing additivity changes little (unlike drifting games)
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  - Game loss at time $t$ is $u_t = r_t$

- ...Solved with a potential $\Phi(u) = \psi \left( \sum_{i=1}^N \phi(u_i) \right)$
  - $\phi$ nonnegative, increasing, twice-diff.
  - $\psi$ concave, nonnegative, strictly increasing, twice-diff.
  - Relaxing additivity changes little \textit{(unlike} drifting games\textit{)}

---
Potential-Based Prediction with Experts

- Prediction with expert advice
  - On iteration $t$, $N$ experts each predict in decision space $Z$
  - Algorithm predicts $z_{A,t} \in Z$, Nature reveals outcome $y_t$
  - Expert $i$ incurs loss $l_{i,t}$, algorithm incurs $l_{A,t}$
  - Instantaneous regret $r_{i,t} = l_{A,t} - l_{i,t}$ to expert $i$

- ...As a game with losses in $\mathbb{R}^N$, one expert per coordinate
  - $r_t \in \mathbb{R}^N$ is vector with components $r_{i,t}$; $R_t = \sum_{i=1}^{t} r_i$
  - Game loss at time $t$ is $u_t = r_t$

- ...Solved with a potential $\Phi(u) = \psi \left( \sum_{i=1}^{N} \phi(u_i) \right)$
  - $\phi$ nonnegative, increasing, twice-diff.
  - $\psi$ concave, nonnegative, strictly increasing, twice-diff.
  - Relaxing additivity changes little (unlike drifting games)
\[ \Phi(R_t) \approx \Phi(R_{t-1}) + \langle \nabla \Phi(R_{t-1}), R_t - R_{t-1} \rangle = \Phi(R_{t-1}) + \langle r_t, \nabla \Phi(R_{t-1}) \rangle \]

- To try to keep \( \Phi(R_t) \) decreasing, control \( \langle r_t, \nabla \Phi(R_{t-1}) \rangle \)
- Generalized Blackwell condition: \( \sup_{y_t \in \mathcal{Y}} \langle r_t, \nabla \Phi(R_{t-1}) \rangle \leq 0 \)

**Theorem**

Let \( C(r_t) = \sup_{u \in \mathbb{R}^N} \psi' \left( \sum_{i=1}^{N} \phi(u_i) \right) \sum_{i=1}^{N} \phi''(u_i) r_{i,t}^2 \). Then for all \( n \geq 1 \),

\[ \Phi(R_n) \leq \Phi(0) + \frac{1}{2} \sum_{t=1}^{n} C(r_t) \]
Generalized Blackwell Condition

Figure 2.1. An illustration of the Blackwell condition with $N = 2$. The dashed line shows the points in regret space with potential equal to 1. The prediction at time $t$ changed the potential from $\Phi(R_{t-1}) = 1$ to $\Phi(R_t) = \Phi(R_{t-1} + r_t)$. Though $\Phi(R_t) > \Phi(R_{t-1})$, the inner product between $r_t$ and the gradient $\nabla \Phi(R_{t-1})$ is negative, and thus the Blackwell condition holds.
Proof of Loss Bound (Potential-Based Forecaster)

Using Taylor’s Theorem and denoting \( \xi \) as some vector \( \in \mathbb{R}^N \),

\[
\Phi(R_t) = \Phi(R_{t-1}) + \langle r_t, \nabla \Phi(R_{t-1}) \rangle + \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \left[ \frac{\partial^2 \Phi}{\partial u_i \partial u_j} \right] \xi r_i, t \ r_j, t
\]

\[
\leq \Phi(R_{t-1}) + \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \left[ \frac{\partial^2 \Phi}{\partial u_i \partial u_j} \right] \xi r_i, t \ r_j, t
\]

\[
\leq \Phi(R_{t-1}) + \frac{1}{2} \left( \psi'' \left( \sum_{i=1}^{N} \phi(\xi_i) \right) \left( \sum_{i=1}^{N} \phi'(\xi_i) r_i, t \right)^2 + \psi' \left( \sum_{i=1}^{N} \phi(\xi_i) \right) \sum_{i=1}^{N} \phi''(\xi_i) r_i, t^2 \right)
\]

Using the concavity of \( \psi \), we therefore have

\[
\Phi(R_t) \leq \Phi(R_{t-1}) + \frac{1}{2} \left( \psi' \left( \sum_{i=1}^{N} \phi(\xi_i) \right) \sum_{i=1}^{N} \phi''(\xi_i) r_i, t^2 \right) \leq \Phi(R_{t-1}) + \frac{1}{2} C(r_t)
\]

Induction then gives the result.
Applications of Potential-Based Prediction

- What algorithms obey Blackwell condition and conditions on $\Phi$?

- Weighted average predictors
  - Predict with a weighted average of experts,
    \[ w_{i,t} \propto \nabla_i \Phi(R_{t-1}) \]
  - Always satisfies Blackwell condition
  - Hedge ($\Phi(u) = \sum_{i=1}^{N} e^{\eta u_i}$), Blackwell’s strategy
    \[ (\Phi(u) = \sum_{i=1}^{N} \left( u_i \right)^2_{\text{+}}) \]
  - Perceptron/Winnow (special mirror descent)
  - Adaboost, polynomial potential, various forms of regret, specialists...
Recap: Potential-Based Approachability

- To try to keep $\Phi(R_t)$ decreasing, control $\langle r_t, \nabla \Phi(R_{t-1}) \rangle$
- Only very relaxed halfspace control possible, so potential can still increase
- But master loss bound is still very useful
Recap: Potential-Based Approachability

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- But master loss bound is still very useful
Blackwell approachability is intimately tied with the question: what can be done by forcing halfspaces?

Drifting games deal with this as well
- Halfspace forcing is a constraint on adversary, by definition satisfying Blackwell condition
- Drifting games set weights = “derivative" of potential
- Boosting, hedging (NormalHedge) are examples

Game-theoretic supermartingales
- Vovk’s algorithms, markets involve forcing a function to lie on a half-line
Approachability Implies No-Regret Strategies

- Potential-based approachability algorithms can be used to play games (experts = finite strategy set)
- Want to keep regrets (payoffs) low, i.e. approach $S = \{ s : s_i \leq 0 \ \forall i \leq N \}$
- $S$ is response-satisfiable (put all weight on best expert) $\implies$ approachable
- So there exists a set of player moves such that

$$
\lim_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} \max_{i \in [N]} r_{i,t} = 0 \implies \lim_{T \to \infty} \max_{i \in [N]} \frac{1}{T} \sum_{t=1}^{T} r_{i,t} = 0
$$

- This verifies the existence of an algorithm with asymptotically vanishing regret - Hannan consistency.
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- This verifies the existence of an algorithm with asymptotically vanishing regret - *Hannan consistency.*
Approachability and No-Regret Strategies

- Approachability games lead to no-regret learning algorithms (potential-based)
- Natural problem considered: online linear optimization (experts setting)
- Generic hammer to apply approachability?
  - Abernethy et al. (2011) produce calibrated probability predictions in $\{0, \frac{1}{m}, \ldots, 1\}$ with it
  - Payoff space $\mathbb{R}^{m+1}$, $\Phi$ measures discrepancy between actual and predicted probabilities for each bin
  - $S$ is a small ball around the origin, response-satisfiable
  - Construction: Halfspace oracle possible to implement efficiently, approachability algorithm: GD
  - Other such examples?
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Blackwell approachability: generalization of minimax to vector-valued games

Can be viewed as minimizing a potential (moving down a conservative force field)

Framework to study halfspace-forcing phenomena in algorithms
Many thanks! Questions?
Sources

- Cesa-Bianchi and Lugosi: Prediction, Learning and Games. 2006. (Ch. 7)