Overview

1 History of Probability Theory
   - Before Kolmogorov
   - During Kolmogorov
   - After Kolmogorov

2 Shafer and Vovk
   - It’s only a game
   - Winning conditions
   - Comparison with measure theory
   - An analogue to variance

3 Efficient Market Hypothesis
   - Securities Market Protocol
This was the day before probability theory was even a field in mathematics, a field without foundations. Pascal and Fermat simply wanted to win a ton of money betting on horses and wanted to first see what it meant for a game to be fair.
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This is what is referred to as *inter alia* (equal terms)

$P[E] = \text{how much money you’re willing to put on a game where you could win 1} \$
Looking at the real world

Bernoulli was the first to suggest that probability can be measured from observation

\[ P\{ |y/N - p| < \epsilon \} > 1 - \delta \]

Now it seems that there could be a more mathematical treatment of probability..
Kolmogorov’s axioms

- The axioms and definitions below relate a set $\Omega$ called the sample space and the set of subsets of $\Omega$, $\mathcal{F}$. Every element in $E \in \mathcal{F}$ is called an event

1. If $E, F \in \mathcal{F}$ then $E \cup F, E \cap F, E \setminus F \in \mathcal{F}$. Or more concisely we say that $\mathcal{F}$ is a field of sets.
2. $\Omega \subset \mathcal{F}$ which with the first axiom means that $\mathcal{F}$ is an algebra of sets
3. Every set $E \in \mathcal{F}$ is assigned a probability which is a non-negative real value using the function $P : E \rightarrow [0, 1]$
4. $P[\Omega] = 1$
5. If $E \cap F = \Phi$ then $P[E \cap F] = P[E] + P[F]$, more generally we get what is called the union bound when $E$ and $F$ are not disjoint then $P[E \cup F] \leq P[E] + P[F]$
6. If $\bigcap_{n=1}^{\infty} E_n = \Phi$ where $E_n \subseteq E_{n-1} \cdots \subseteq E_1$ we have that $\lim_{n \to \infty} P[E_n] = 0$. This axiom with axiom 2 allows us to call $\mathcal{F}$ a $\sigma$-algebra

- A random variable $x$ is then understood as a mapping from the size of elements of $\mathcal{F}$ with respect to the probability measure $P$
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Definition (ZFC Axiom of Choice)
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Some Set Theory

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Every 2 player game with perfect information where two players pick natural numbers at every turn is already determined.
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Weird things also happen: We get that there is no such thing as non-measurable sets
Von Mises was the first to propose that probability could find its foundations in games.

Given a bit string 001111

Predict the odds of a 1 (number shouldn’t change much if we look at a subsequence called a collective)
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Given a bit string 001111

Predict the odds of a 1 (number shouldn’t change much if we look at a subsequence called a collective)

Fortunately we have a method of quantifying how difficult it is to predict the next bit in a string: Kolmogorov complexity!
Martingales

Originally Martingales are a gambling strategy that can guarantee a win of 1$ given an infinite supply of money

\[ \alpha + 2\alpha + \cdots + 2^i \alpha \]

stop as soon as you win once
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Definition (Martingale)

Given a sequence of outcomes \( x_1, \ldots, x_n \) we call a capital process \( L \) if

\[ E[L(x_1, \ldots, x_n) \mid x_1, \ldots, x_{n-1}] = L(x_1, \ldots, x_n) \]
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\( L(E) \to \infty \) if \( E \) has probability 0 (more on this next slide)

Now we define the probability of an event \( E \) as

\[ P(E) = \inf \{ L_0 \mid \lim_{n \to \infty} L_n \geq I \} \]
Martingales

Theorem (Doob’s inequality)

\[ P\left[ \sup_{n} L(x_1, \ldots, x_n) \geq \lambda \right] \leq \frac{1}{\lambda} \]

Look familiar?
Martingales

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Markov’s inequality!

\[ P[x \geq \lambda] \leq \frac{E_x}{\lambda} \]
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Other Chernoff bounds can be derived in this way as well.
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- $K_i$ is the skeptic’s capital at time $i$
- $M_n$ is the amount of tickets that the skeptic purchases
- $x_n$ is the value of a ticket (determined by nature)
Theorem

There exists a winning strategy for skeptic but let’s formally define what we mean by winning

\[ \mathcal{K}_0 = 1. \]

FOR \( n = 1, 2, \ldots: \)

Skeptic announces \( M_n \in \mathbb{R}. \)
Reality announces \( x_n \in \{-1, 1\}. \)

\[ \mathcal{K}_n := \mathcal{K}_{n-1} + M_n x_n. \]
Winning conditions

We claim that the skeptic wins if $K_n > 0 \forall n$ and if one two things happen, either

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} x_i = 0$$

The first condition has two different interpretations in the finance literature. It is called a self-financing strategy because it means that the skeptic can remain in the game forever without ever having to borrow money. Also, it means that the skeptic wins if nature is forced to play randomly.

The second makes sense: you win if you become infinitely rich, but since this is unlikely, this condition embodies the infinitary hypothesis which says that there is no strategy that avoids bankruptcy that guarantees that the skeptic becomes infinitely rich.

If $P(E) = 0$ then that means that the skeptic can become infinitely rich if $E$ happens.
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**Law of Large Numbers.**

Skeptic bets $\epsilon$ on heads, this forces nature not to play heads often or else skeptic will become infinitely rich. So nature will start playing tails, when that happens skeptic puts an $\epsilon$ on tails.
What if $x_n \in [-1, 1]$ instead of $\{-1, 1\}$?

Players: Skeptic, Reality

Protocol:

$K_0 = 1.$

FOR $n = 1, 2, \ldots$:

- Skeptic announces $M_n \in \mathbb{R}$.
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We will need some terminology to tackle this problem we define a real valued function on $\Omega$ called $P$ which is a strategy that takes situations $s = x_1, x_2, \ldots, x_n$ and decides the number of tickets to buy $P(s)$. 
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\[
K^P(x_1x_2 \ldots x_n) = K^P(x_1x_2 \ldots x_{n-1}) + P(x_1x_2 \ldots x_n)x_n
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Proof of Bounded Fair Coin Game

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Definition

Skeptic forces an event $E$ if $K^P(s) = \infty \forall s \in E^c$
Lemma

The skeptic can force

\[ \limsup_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} x_i \leq \epsilon \]

and

\[ \limsup_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} x_i \geq \epsilon \]
Proof of Bounded Fair Coin Game

Proof.

Take 1 as the starting capital

$$1 + K^P(x_1x_2 \ldots x_n) = (1 + K^P(x_1 \ldots x_{n-1}))(1 + \epsilon x_n) = \prod_{i=1}^{n}(1 + \epsilon x_i) < C$$

Where $C$ is a constant so take the log on both sides

$$\sum_{i=1}^{n} \ln(1 + \epsilon x_i) \leq D$$

Now use $\ln(1 + t) \geq t - t^2$ when $t \geq -1/2$

$$\frac{1}{n} \sum_{i=1}^{n} x_i \leq \frac{D}{\epsilon n} + \epsilon$$

And we get the top part of the lemma. Replace by $-\epsilon$ to get the second
Somebody has got to be setting the prices, let a forecaster announce price of ticket at iteration $n$ as $m_n$

**Parameter:** $C > 0$

**Players:** Forecaster, Skeptic, Reality

**Protocol:**

\[
\mathcal{K}_0 := 1.
\]

FOR $n = 1, 2, \ldots$

- Forecaster announces $m_n \in [-C, C]$.
- Skeptic announces $M_n \in \mathbb{R}$.
- Reality announces $x_n \in [-C, C]$.
- $\mathcal{K}_n := \mathcal{K}_{n-1} + M_n(x_n - m_n)$.

**Theorem**

There exists a winning strategy for skeptic by reduction to the bounded fair coin game.
Bounded Forecast games

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**Theorem**

*There exists a winning strategy for skeptic by reduction to the bounded fair coin game.*

**Proof.**

First divide all prices by $C$ to normalize prices to $[-1, 1]$ then set $m_n = 0$ and we recover the previous game. Note we also need to change the first condition to $\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} (x_i - m_i) = 0$.
Assuming $X_i$ are i.i.d random variables with mean $\mu$ and variance $\sigma^2$ we define $A_n = \frac{X_1 + X_2 + \cdots + X_n}{n}$ then $E[A_n] = \frac{n\mu}{n} = \mu$ and similarly $\text{Var}[A_n] = \frac{n\sigma^2}{n^2} = \frac{\sigma^2}{n}$. By Chebyshev’s inequality we get the weak law of large numbers

$$P( |A_n - \mu| \geq \epsilon) \leq \frac{\text{Var}[A_n]}{\epsilon^2} = \frac{\sigma^2}{n\epsilon^2}$$
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To prove Chebyshev’s we define $A = \{ w \in \Omega \mid |X(w)| \geq \alpha \}$. $X(w) \geq \alpha I_A(w)$. Take the expectation on both sides to get

$E(|X|) \geq \alpha E(I_A) = \alpha P(A)$
Measure theoretic law of large numbers

Assuming $X_i$ are i.i.d random variables with mean $\mu$ and variance $\sigma^2$ we define $A_n = \frac{X_1 + X_2 + \cdots + X_n}{n}$ then $E[A_n] = \frac{n\mu}{n} = \mu$ and similarly $\text{Var}[A_n] = \frac{n\sigma^2}{n^2} = \sigma^2/n$. By Chebyshev’s inequality we get the weak law of large numbers

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In game theoretic proof we don’t need i.i.d assumption
Measure theoretic law of large numbers

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In game theoretic proof we don’t need i.i.d assumption we don’t even to assume a distribution exists!
Unbounded game

Players: Forecaster, Skeptic, Reality

Protocol:

\[ \mathcal{K}_0 := 1. \]

FOR \( n = 1, 2, \ldots : \)

Forecaster announces \( m_n \in \mathbb{R} \) and \( v_n \geq 0 \).
Skeptic announces \( M_n \in \mathbb{R} \) and \( V_n \geq 0 \).
Reality announces \( x_n \in \mathbb{R} \).

\[ \mathcal{K}_n := \mathcal{K}_{n-1} + M_n (x_n - m_n) + V_n ((x_n - m_n)^2 - v_n). \]
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Theorem

If \( \sum_{n=1}^{\infty} \frac{V_n}{n^2} < \infty \) then the skeptic has a winning strategy
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Theorem

If \[ \sum_{n=1}^{\infty} \frac{v_n}{n^2} < \infty \] then the skeptic has a winning strategy

Proof.

Similar in nature to proof of the bounded fair coin game. Main idea is that the skeptic’s capital is a supermartingale (a sequence that decreases in expectation)
What about an application

Suppose you’re a clever young guy/gal who wants to make money off of these ideas
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A natural next step is to make an infinitely large amount of money off the stock market
Efficient Market Hypothesis

Unfortunately it seems that its difficult to have consistently better returns than the market and we will prove this. We make two assumptions that transaction costs are negligible (not as controversial as it sounds) and that the capital of a specific investor isn’t too big relative to the market.
**Parameters:** $\mathcal{K}_0 > 0$, natural number $K > 1$

**Players:** Opening Market, Investor, Skeptic, Closing Market

**Protocol:**

FOR $n = 1, 2, \ldots$:
- Opening Market selects $m_n \in [0, 1]^K$ such that $\sum_{k=1}^{K} m_{n}^{k} = 1$.
- Investor selects $g_n \in \mathbb{R}^K$.
- Skeptic selects $h_n \in \mathbb{R}^K$.
- Closing Market selects $x_n \in [-1, \infty)^K$ such that $m_n \cdot x_n = 0$.
- $\mathcal{K}_n := \mathcal{K}_{n-1} + h_n \cdot x_n$. 
**Parameters:** $\mathcal{K}_0 > 0$, natural number $K > 1$

**Players:** Opening Market, Investor, Skeptic, Closing Market

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FOR $n = 1, 2, \ldots$

Opening Market selects $m_n \in [0, 1]^K$ such that $\sum_{k=1}^{K} m_n^k = 1$.

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**Proof.**

Maybe next time, Finance theory might need its own talk :)
Shafer and Vovk (2001)
Probability and Finance It’s only a Game!

Ramon Van Handel
Stochastic Calculus

Peter Clark
All I ever needed to know from Set Theory
Let’s think about how this could change machine learning, talk to me and let’s write a paper about it!