’t Hooft anomalies and higher form symmetries

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The 1-form symmetry of abelian Spin-Chern-Simons

We consider abelian Spin-Chern-Simons theory for a $U(1)$ gauge connection A on a 3-manifold M_3 with action

$$S_{\mathrm{CS}}(\kappa) = \frac{\kappa}{4\pi}\int_{M_3} A\mathrm{d}A = 2\pi\kappa\int_{M_4}\frac{c_1^2}{2}, \qquad \partial M_4 = M_3 \tag{1}$$

The level $\kappa \in \mathbb{Z}$ is quantized by integers in order for the exponential of the action $\mathrm{e}^{\mathrm{i}S_{\mathrm{CS}}(\kappa)}$ to be gauge invariant.

Remark. Here we use the fact that every orientable 3-manifold is spin and that $\Omega_3^{Spin}(BU(1)) \cong 0$ so that we can define the action (1) as the integral of the class $\frac{c_1^2}{2}$ on a spin 4-manifold whose boundary is M_3. Integrality of this class is guaranteed by the fact that the intersection form on a spin 4-manifold is even. The quadratic refinement of the intersection form then depends on the choice of spin structure on the 4-manifold. The CS partition function then must depend on the restriction of this spin structure on M_3, therefore the theory is called *Spin-Chern-Simons* (or *fermionic* CS).

The space of $U(1)$ gauge fields A has an action by the group of flat $U(1)$ connections ϵ defined by the shift

$$A \mapsto A + \epsilon, \qquad \frac{1}{2\pi}\epsilon \in H^1(M_3, \mathbb{R}/\mathbb{Z}) \tag{2}$$

Observe that ϵ is NOT a gauge transformation of A because in general it is not exact. A gauge transformation would act as $A \mapsto A + g^{-1}\mathrm{d}g$ for $g \in \mathrm{Hom}(M_3, U(1))$.

The function $\mathrm{e}^{\mathrm{i}S_{\mathrm{CS}}(\kappa)}$ however is not invariant under the action of this group of transformations, but it is invariant under the subgroup of flat connections with $\mathbb{Z}_\kappa$ holonomy

$$A \mapsto A + \frac{1}{\kappa}\epsilon, \qquad \frac{1}{2\pi}\epsilon \in H^1(M_3, \mathbb{Z}/\kappa\mathbb{Z}) \tag{3}$$

$$\delta_\epsilon S_{\mathrm{CS}}(\kappa) = \frac{1}{2\pi}\int_{M_3}\epsilon\,\mathrm{d}A \in 2\pi\mathbb{Z} \tag{4}$$

Remark. The idea here is that the transformation in (3) changes both the connection A and the associated principal bundle, so that if we gauge this group action we are identifying different $U(1)$-bundles whose characteristic classes differ by a $\mathbb{Z}_\kappa$-torsion element in $H^2(M_3, \mathbb{Z})$.

Reduction to 2d

Following [Kapustin,Seiberg] if we assume that $M_3 = M_2 \times \mathbb{R}$ we can perform a dimensional reduction of the theory in (1) to the 2-manifold M_2. The dimensionally reduced action is

$$S_{2d}(\kappa) = \frac{\kappa}{2\pi} \int_{M_2} A_3 F \tag{5}$$

where:

- $F \in \Omega^2(M_2)$ is the curvature of the reduced connection;
- $A_3 \in \Omega^0(M_2)$ is the component of the 3d connection along the $\mathbb{R}$ direction.

The original 1-form $\mathbb{Z}_\kappa$ symmetry now acts as an ordinary 0-form symmetry

$$\phi := \mathrm{e}^{2\pi \mathrm{i} A_3} \mapsto \mathrm{e}^{2\pi \mathrm{i} \frac{1}{\kappa} \epsilon} \phi \tag{6}$$

by shifting the phase of the compact scalar $\phi \in \mathrm{Hom}(M_2, S^1)$.

The symmetry operators

The objects charged under the 1-form symmetry are the Wilson loop operators

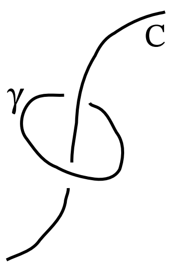
$$U_m(C) := \exp\left(\mathrm{i}m \int_C A\right) \tag{7}$$

while the generators of the 1-form symmetry are themselves 1-dimensional extended objects

$$U_n(\gamma) := \exp\left(\mathrm{i}n \int_\gamma A\right), \qquad n \in \mathbb{Z}_\kappa \tag{8}$$

with the action specified by

$$\begin{aligned} U_n(\gamma) U_m(C) U_n^{-1}(\gamma) &= \exp\left(\mathrm{i}m \int_C (A + \frac{n}{\kappa}\epsilon)\right) \\ &= \exp\left(2\pi \mathrm{i} \frac{mn}{\kappa} \mathrm{lk}(\gamma, C)\right) U_m(C) \end{aligned} \tag{9}$$



Here ϵ can be identified with the "dual" of the loop γ as defined by the relation

$$\frac{1}{2\pi} \int_C \epsilon = \mathrm{lk}(\gamma, C) \mod \kappa \tag{10}$$

One way to obtain the commutation relation (9) is via a path integral argument. The expectation value of a commutator of two operators is given by the Green function of the kinetic operator in the action. In this case the Spin-CS action is Gaussian and the kinetic operator is the exterior derivative, hence

$$\begin{aligned} \langle 0 | [\mathrm{i} \int_\gamma A, \mathrm{i} \int_C A] | 0 \rangle &= - \int_\gamma \mathrm{d}x^\mu \int_C \mathrm{d}y^\nu \langle A_\mu(x) A_\nu(y) \rangle \\ &= \frac{2\pi \mathrm{i}}{\kappa} \int_\gamma \int_C \frac{1}{4\pi} \varepsilon_{\mu\nu\rho} \frac{(x-y)^\rho \mathrm{d}x^\mu \mathrm{d}y^\nu}{2! |x-y|^3} \\ &= \frac{2\pi \mathrm{i}}{\kappa} \mathrm{lk}(\gamma, C) \end{aligned} \tag{11}$$

As a consequence we have that the generators of the group are themselves charged under the symmetry. In other words, the generators of the symmetry do not commute (even though the group is abelian).

More precisely, we define $\mathbb{G}$ to be the symmetry group of the theory. Then the commutation relation (9) implies that the quantum Hilbert space provides a projective representation of $\mathbb{G}$, i.e. a representation for its central extension

$$1 \to \mathbb{Z}_\kappa \to \widehat{\mathbb{G}} \to \mathbb{G} \to 1 \tag{12}$$

which is classified by the group cohomology cocycle

$$\mu = \frac{1}{\kappa}\mathrm{lk}(-,-) \mod 1, \qquad \mu \in H^2(B\mathbb{G}, \mathbb{Z}_\kappa) \tag{13}$$

Roughly speaking, one should identify $\mathbb{G}$ with the (abelian) group $H^1(M_3, \mathbb{Z}_\kappa)$.

The fact that the symmetry is only realized projectively is the sign of an **anomaly**.

An anomaly for a global symmetry is usually called a **'t Hooft anomaly**. The theory is still well defined but the symmetry cannot be gauged.

't Hooft anomaly

Another way to identify the presence of an anomaly for the $\mathbb{Z}_k$ 1-form symmetry is by gauging it and observing that the partition function is not well defined.

Gauging the $\mathbb{Z}_\kappa$ symmetry corresponds to summing over all insertions of operators $U_n(\gamma)$ in the path integral. But because of the phase appearing in the commutation relations (9), the path integral sums to zero.

To be more precise, the gauging of the 1-form symmetry corresponds to quotienting the $U(1)$ gauge group associated to A by its $\mathbb{Z}_\kappa$ subgroup

$$\begin{array}{ccc} \mathbb{Z}_\kappa & \to & U(1) \\ \downarrow & & \downarrow^{\theta_\kappa} \\ * & \to & U(1)/\mathbb{Z}_\kappa \end{array} \tag{14}$$

The map θ_κ induces a multiplication by κ in cohomology

$$H^2(M_3, \mathbb{Z}) \xrightarrow{\theta_\kappa} H^2(M_3, \mathbb{Z}) \tag{15}$$

so that a gauge field A in the $U(1)$ theory descends to a well defined $U(1)/\mathbb{Z}_\kappa$ gauge field in the quotient theory iff the class of its curvature is in the image of θ_κ, i.e. iff

$$\frac{\mathrm{d}A}{2\pi} \in \kappa\mathbb{Z} \tag{16}$$

We define $a = \theta_\kappa^{-1} A = \frac{1}{\kappa} A$ the pre-image of a generic $U(1)$ auge field. Then the action for a is

$$S_{\mathrm{CS}//\mathbb{Z}_\kappa}(\kappa) = \frac{\kappa}{4\pi}\int_{M_3} a\mathrm{d}a = \frac{1}{4\pi\kappa}\int_{M_3} A\mathrm{d}A = S_{\mathrm{CS}}(1/\kappa) \tag{17}$$

which is clearly not well defined.

Either:

- the CS level is integer but a is not properly quantized,
- or A is properly quantized but the CS level is fractional

hence the partition function is not gauge invariant.

A non-abelian example

Let us consider now the CS theory of an $SU(2)$ gauge field at level κ.

The global 1-form symmetry is the one associated to the center of the gauge group

$$Z(SU(2)) \cong \mathbb{Z}_2 \tag{18}$$

The generators of the 1-form symmetry are the Wilson loops in the $SU(2)$ representation of spin $j = \kappa/2$ and they act on all other Wilson loops as

$$U_{\frac{\kappa}{2}}(\gamma)V_j(C)U^{-1}_{\frac{\kappa}{2}}(\gamma) = (-1)^{2j\,\mathrm{lk}(\gamma,C)}V_j(C), \qquad m \in \mathbb{Z}_2 \tag{19}$$

The symmetry *should* also be implemented by the shift

$$A \mapsto A + \frac{1}{2}\epsilon\,\mathbb{I}, \qquad \frac{1}{2\pi}\epsilon \in H^1(M_3, \mathbb{Z}_2) \tag{20}$$

so that

$$\frac{1}{2\pi}\int_C \epsilon = \mathrm{lk}(\gamma, C) \mod 2 \tag{21}$$

Similarly to the previous case, if we gauge this 1-form symmetry we end up with CS theory with gauge group $SO(3) \cong SU(2)/\mathbb{Z}_2$. What is the relation between the levels of the two theories? The answer is

$$S^{SU(2)}_{\mathrm{CS}//\mathbb{Z}_2}(\kappa) = S^{SO(3)}_{\mathrm{CS}}(\kappa/2) \tag{22}$$

One way to see this is by observing that the action $S^{SO(3)}_{\mathrm{CS}}$ can be defined as

$$S^{SO(3)}_{\mathrm{CS}}(\kappa) = 2\pi\kappa\int_{M_4} p_1 \tag{23}$$

where $\partial M_4 = M_3$ and p_1 is the first Pontryagin class, generator of $H^4(BSO(3), \mathbb{Z})$.

For a principal $SO(3)$-bundle we have the identity

$$p_1 = w_2 \cup w_2 \mod 2 \tag{24}$$

If the bundle can be lifted to an $SU(2)$-bundle then $w_2 = 0$ which means that

$$\frac{p_1}{2} = \lambda \tag{25}$$

is a well defined integral class. But λ can be identified with the generator of $H^4(BSU(2), \mathbb{Z})$, which is used to define the action of $SU(2)$ CS theory

$$S^{SU(2)}_{\mathrm{CS}}(\kappa) = 2\pi\kappa\int_{M_4}\lambda = 2\pi\kappa\int_{M_4}\frac{p_1}{2} = S^{SO(3)}_{\mathrm{CS}}(\kappa/2) \tag{26}$$

If the $SU(2)$ theory is well defined, then gauging the $\mathbb{Z}_2$ 1-form symmetry leads to an $SO(3)$ with half the level. If the original level is odd the quotient theory in not well defined.

In this case we say that the $SU(2)$ CS theory with **odd level** has an **'t Hooft anomaly** for the 1-form symmetry.

Similarly one can work out the generalization to $SU(N)$ CS theory with $\mathbb{Z}_N$ 1-form symmetry.

References

- Kapustin and Seiberg, *Coupling a QFT to a TQFT and Duality*, [arXiv:1401.0740].
- Gaiotto, Kapustin, Seiberg and Willett, *Generalized Global Symmetries*, [arXiv:1412.5148].