’t Hooft anomalies and higher form symmetries

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1 The 1-form symmetry of abelian Spin-Chern-Simons

Following Sec 4.4 of [1] we consider abelian Spin-Chern-Simons theory for a $U(1)$ gauge connection A on a 3-manifold M_3 with action

$$S_{\mathrm{CS}}(\kappa) = \frac{\kappa}{4\pi}\int_{M_3} A\mathrm{d}A = 2\pi\kappa\int_{M_4}\frac{c_1^2}{2}, \qquad \partial M_4 = M_3 \tag{1}$$

The level $\kappa \in \mathbb{Z}$ is quantized by integers in order for the exponential of the action $\mathrm{e}^{\mathrm{i}S_{\mathrm{CS}}(\kappa)}$ to be gauge invariant.

> **Remark**. Here we use the fact that every orientable 3-manifold is spin and that $\Omega_3^{Spin}(BU(1)) \cong 0$ so that we can define the action (1) as the integral of the class $\frac{c_1^2}{2}$ on a spin 4-manifold whose boundary is M_3. Integrality of this class is guaranteed by the fact that the intersection form on a spin 4-manifold is even. The quadratic refinement of the intersection form then depends on the choice of spin structure on the 4-manifold. The CS partition function then must depend on the restriction of this spin structure on M_3, therefore the theory is called *Spin-Chern-Simons* (or *fermionic* CS).

The space of $U(1)$ gauge fields A has an action by the group of flat $U(1)$ connections ϵ defined by the shift

$$A \mapsto A + \epsilon, \qquad \frac{1}{2\pi}\epsilon \in H^1(M_3, \mathbb{R}/\mathbb{Z}) \tag{2}$$

This action modifies the holonomy of the connection but not the curvature, therefore it is a symmetry of the Yang-Mills functional and it naturally acts on Wilson loop operators.

Observe that ϵ is NOT a gauge transformation of A. A gauge transformation would act as $A \mapsto A + g^{-1}\mathrm{d}g$ for $g \in \mathrm{Hom}(M_3, U(1))$, so that $g^{-1}\mathrm{d}g$ is flat but its periods are integers and therefore do not change the holonomy of A.

The function $e^{iS_{CS}(\kappa)}$ however is not invariant under the action of this group of transformations, but it is invariant under the subgroup of flat connections with $\mathbb{Z}_\kappa$ holonomy

$$A \mapsto A + \frac{1}{\kappa}\epsilon, \qquad \frac{1}{2\pi}\epsilon \in H^1(M_3, \mathbb{Z}/\kappa\mathbb{Z}) \tag{3}$$

$$\delta_\epsilon S_{CS}(\kappa) = \frac{1}{2\pi}\int_{M_3} \epsilon\, dA \in 2\pi\mathbb{Z} \tag{4}$$

> **Remark**. The idea here is that the transformation in (3) changes both the connection A and the associated principal bundle, so that if we gauge this group action we are identifying different $U(1)$-bundles whose characteristic classes differ by a $\mathbb{Z}_\kappa$-torsion element in $H^2(M_3, \mathbb{Z})$, i.e. those elements that are in the image of the Bockstein homomorphism
>
> $$\underbrace{H^1(M_3, \mathbb{Z}_\kappa)}_{\text{flat } \mathbb{Z}_\kappa \text{ connections}} \hookrightarrow \underbrace{H^1(M_3, \mathbb{R}/\mathbb{Z})}_{\text{flat } U(1) \text{ connections}} \xrightarrow{\beta} \underbrace{H^2(M_3, \mathbb{Z})}_{\substack{\text{characteristic class} \\ \text{of the } U(1)\text{-bundle}}} \tag{5}$$

2 Reduction to 2d

Following [2] if we assume that $M_3 = M_2 \times \mathbb{R}$ we can perform a dimensional reduction of the theory in (1) to the 2-manifold M_2. The dimensionally reduced action is

$$S_{2d}(\kappa) = \frac{\kappa}{2\pi}\int_{M_2} A_3 F \tag{6}$$

where:

- $F \in \Omega^2(M_2)$ is the curvature of the reduced connection;
- $A_3 \in \Omega^0(M_2)$ is the component of the 3d connection along the $\mathbb{R}$ direction.

The original 1-form $\mathbb{Z}_\kappa$ symmetry now acts as an ordinary 0-form symmetry

$$\phi := e^{2\pi i A_3} \mapsto e^{2\pi i \frac{1}{\kappa}\epsilon}\phi \tag{7}$$

by shifting the phase of the compact scalar $\phi \in \mathrm{Hom}(M_2, S^1)$, with $\epsilon \in \mathbb{Z}_\kappa$.

3 The symmetry operators

The objects charged under the 1-form symmetry are the Wilson loop operators

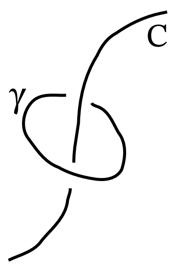
$$U_m(C) := \exp\left(im\int_C A\right) \tag{8}$$

while the generators of the 1-form symmetry are themselves 1-dimensional extended objects

$$U_n(\gamma) := \exp\left(in\int_\gamma A\right), \qquad U_n(\gamma)^\kappa = 1 \tag{9}$$

with the action specified by

$$\begin{aligned} U_n(\gamma)U_m(C)U_n^{-1}(\gamma) &= \exp\left(im\int_C (A + \frac{n}{\kappa}\epsilon)\right) \\ &= \exp\left(2\pi i\frac{mn}{\kappa}\mathrm{lk}(\gamma, C)\right)U_m(C) \end{aligned} \tag{10}$$



Here ϵ can be identified with the "dual" of the loop γ as defined by the relation

$$\frac{1}{2\pi}\int_C \epsilon = \mathrm{lk}(\gamma, C) \mod \kappa \tag{11}$$

One way to obtain the commutation relation (10) is via a path integral argument. The expectation value of a commutator of two operators is given by the Green function of the kinetic operator in the action. In this case the Spin-CS action is Gaussian and the kinetic operator is the exterior derivative, hence

$$\begin{aligned}\langle 0|[\mathrm{i}\int_\gamma \widehat{A}, \mathrm{i}\int_C \widehat{A}]|0\rangle &= -\int_\gamma \mathrm{d}x^\mu \int_C \mathrm{d}y^\nu \langle A_\mu(x) A_\nu(y)\rangle \\ &= \frac{2\pi\mathrm{i}}{\kappa}\int_\gamma\int_C \frac{1}{4\pi}\varepsilon_{\mu\nu\rho}\frac{(x-y)^\rho \mathrm{d}x^\mu \mathrm{d}y^\nu}{2!|x-y|^3} \\ &= \frac{2\pi\mathrm{i}}{\kappa}\mathrm{lk}(\gamma, C)\end{aligned} \tag{12}$$

Equation (10) can be equivalently rewritten as

$$\left\langle \mathrm{e}^{\mathrm{i}n\int_\gamma A}\mathrm{e}^{\mathrm{i}m\int_C A}\right\rangle = \exp\left(2\pi\mathrm{i}\frac{mn}{\kappa}\mathrm{lk}(\gamma, C)\right) \tag{13}$$

where the phase in the r.h.s. can be interpreted as a Schwinger term.

Remark. The Gauss formula for the linking number only makes sense for $M_3 \cong \mathbb{R}^3$. For a generic 3-manifold M_3 we give the two following descriptions for computing the commutation relations of the symmetry algebra.

Lagrangian picture. To each symmetry generator there is associated a flat connection ϵ inducing the transformation (3). Given two such transformations $\epsilon_{1,2}$, the commutator of the corresponding operators is computed as

$$[U(\epsilon_1), U(\epsilon_2)] = \exp\left(\frac{2\pi\mathrm{i}}{\kappa}\int_{M_3}\epsilon_1 \cup \beta\epsilon_2\right) \tag{14}$$

which is defined through the **link pairing** (or **torsion pairing**) [3]

$$\mathrm{lk} := \int_{M_3}(-)\cup\beta(-) : H^1(M_3, \mathbb{Z}_\kappa)\times H^1(M_3,\mathbb{Z}_\kappa)\to\mathbb{Z}_\kappa \tag{15}$$

More specifically, let $\gamma_{1,2} \in \mathrm{Tors}H_1(M_3)$ such that

$$\mathrm{PD}(\gamma_i) = \beta(\epsilon_i) \tag{16}$$

and find $\Gamma_2 \in H_2(M_3)$ such that $\partial\Gamma_2 = \kappa\gamma_2$, then

$$\mathrm{lk}(\epsilon_1, \epsilon_2) = \gamma_1\cdot\Gamma_2 \mod \kappa \tag{17}$$

Hamiltonian picture. We perform canonical quantization by assuming $M_3 = \Sigma\times\mathbb{R}_{\mathrm{time}}$. On the Hilbert space $\mathcal{H}_\Sigma$ we have the action of the operators

$$\widehat{U}(\gamma) = \exp\left(\mathrm{i}\int_\gamma \widehat{A}\right), \qquad \gamma\in H_1(\Sigma,\mathbb{Z}) \tag{18}$$

Given a symplectic basis $\gamma_{i=1,\dots,2g}$ of 1-cycles in Σ we can define canonical *coordinates* and *momenta* as

$$\hat{a}_i = -\mathrm{i}\log\widehat{U}(\gamma_i) \tag{19}$$

and the canonical commutation relations are specified via the symplectic form on $H_1(\Sigma,\mathbb{Z})$, i.e.

$$[\hat{a}_i, \hat{a}_j] = -\frac{2\pi\mathrm{i}}{\kappa}\omega_{ij} \tag{20}$$

where ω_{ij} is the oriented intersection number of γ_i and γ_j [4].

As a consequence we have that the generators of the group are themselves charged under the symmetry. In other words, the generators of the symmetry do not commute (even though the group is abelian).

More precisely, we define $\mathbb{G}$ to be the symmetry group of the theory. Then the commutation relation (10) implies that the quantum Hilbert space provides a projective representation of $\mathbb{G}$, i.e. a representation for its (non-split) central extension

$$1 \to \mathbb{Z}_\kappa \to \widehat{\mathbb{G}} \to \mathbb{G} \to 1 \tag{21}$$

which is classified by the group cohomology cocycle

$$\mu = \frac{1}{\kappa}\mathrm{lk}(-,-) \mod 1, \qquad \mu \in H^2(B\mathbb{G}, \mathbb{Z}_\kappa) \tag{22}$$

Roughly speaking, one should identify $\mathbb{G}$ with the (abelian) group $H^1(M_3, \mathbb{Z}_\kappa)$, or better yet the Poincare' dual of its image under the Bockstein.

> **Remark**. In the Hamiltonian formulation, $\widehat{\mathbb{G}} \cong \mathrm{Heis}(H_1(\Sigma), \mathbb{Z}_\kappa)$ and via Stone-von Neumann we find
> $$\mathcal{H}_\Sigma \cong \underbrace{\mathbb{C}^\kappa \otimes \cdots \otimes \mathbb{C}^\kappa}_{g} \tag{23}$$

The fact that the symmetry is only realized projectively is the sign of an **anomaly**. An anomaly for a global symmetry is usually called a **'t Hooft anomaly**. The theory is still well defined but the symmetry cannot be gauged.

This is due to the fact that in a gauge theory the vacuum must be invariant under the full group of gauge symmetries. In the case that the Hilbert space is a projective representation, this cannot be the case because the action of the central element is given by a non-zero C-number (complex phase).

4 The 't Hooft anomaly

Another way to identify the presence of an anomaly for the $\mathbb{Z}_k$ 1-form symmetry is by gauging it and observing that the partition function is not well defined.

Gauging the $\mathbb{Z}_\kappa$ symmetry corresponds to summing over all insertions of operators $U_n(\gamma)$ in the path integral. But because of the phase appearing in the commutation relations (10), the path integral sums to zero.

To be more precise, the gauging of the 1-form symmetry corresponds to quotienting the $U(1)$ gauge group associated to A by its $\mathbb{Z}_\kappa$ subgroup

$$\begin{array}{ccc} \mathbb{Z}_\kappa & \to & U(1) \\ \downarrow & & \downarrow^{\cdot\kappa} \\ * & \to & U(1)/\mathbb{Z}_\kappa \end{array} \tag{24}$$

The quotient map induces a multiplication by κ in cohomology

$$H^2(M_3, \mathbb{Z}) \xrightarrow{\cdot\kappa} H^2(M_3, \mathbb{Z}) \tag{25}$$

so that a gauge field A in the $U(1)$ theory descends to a well defined $U(1)/\mathbb{Z}_\kappa$ gauge field in the quotient theory iff the class of its curvature is a multiple of κ, i.e. iff

$$\int \frac{\mathrm{d}A}{2\pi} \in \kappa\mathbb{Z} \tag{26}$$

We define $a = \frac{1}{\kappa}A$ the pre-image of a generic $U(1)$ gauge field. Then the action for a is

$$S_{\mathrm{CS}//\mathbb{Z}_\kappa}(\kappa) = \frac{\kappa}{4\pi}\int_{M_3} a\mathrm{d}a = \frac{1}{4\pi\kappa}\int_{M_3} A\mathrm{d}A = S_{\mathrm{CS}}(1/\kappa) \tag{27}$$

which is clearly not well defined.

Either:

- the CS level is integer but a is not properly quantized,
- or A is properly quantized but the CS level is fractional

hence the exponential of the action is not gauge invariant.

5 The gauged theory

We consider now the theory in which the 1-form symmetry is also gauged. We couple the 1-form symmetry to a background field B, which is a class in $H^2(M_3, \mathbb{Z}_\kappa)$.

If B is non-trivial then A is not a connection on a $U(1)$-bundle. It is instead a connection on a **twisted** $U(1)$-bundle, with twist specified by the cohomology class of B.

The curvature of A has fractional periods

$$c_1(F_A) = \frac{1}{\kappa} B \mod 1 \tag{28}$$

Equivalently, the transition functions of the bundle satisfy

$$g_{jk}\, g_{ij} = \mathrm{e}^{\frac{2\pi \mathrm{i}}{\kappa} B_{ijk}}\, g_{ik} \tag{29}$$

The anomaly of the CS theory can be expressed through inflow from a 4d bulk term [5]

$$S_{\text{anomaly}} = 2\pi \frac{1}{\kappa} \int_{M_4} \frac{\mathfrak{P}_\kappa(B)}{2} \tag{30}$$

where $\mathfrak{P}_\kappa : H^2(M_4, \mathbb{Z}_\kappa) \to H^4(M_4, \Gamma(\mathbb{Z}_\kappa))$ is the Pontryagin square operation [6].

The anomaly can also be interpreted as a 4d discrete theta-term of the type described in [7].

6 A non-abelian example

Let us consider now the CS theory of an $SU(2)$ gauge field at level κ.

The global 1-form symmetry is the one associated to the center of the gauge group

$$Z(SU(2)) \cong \mathbb{Z}_2 \tag{31}$$

The symmetry is implemented by the shift

$$A \mapsto A + \frac{1}{2}\epsilon\, \mathbb{I}, \qquad \frac{1}{2\pi}\epsilon \in H^1(M_3, \mathbb{Z}_2) \tag{32}$$

The generators of the 1-form symmetry are the Wilson loops in the $SU(2)$ representation of spin $j = \kappa/2$ and they act on all other Wilson loops as

$$U_{\frac{\kappa}{2}}(\gamma) V_j(C) U^{-1}_{\frac{\kappa}{2}}(\gamma) = (-1)^{2j\, \mathrm{lk}(\gamma, C)} V_j(C) \tag{33}$$

Similarly to the previous case, if we gauge this 1-form symmetry we end up with CS theory with gauge group $SO(3) \cong SU(2)/\mathbb{Z}_2$. What is the relation between the levels of the two theories? The answer is

$$S^{SU(2)}_{\mathrm{CS}//\mathbb{Z}_2}(\kappa) = S^{SO(3)}_{\mathrm{CS}}(\kappa/2) \tag{34}$$

One way to see this is by observing that the action $S^{SO(3)}_{\mathrm{CS}}$ can be defined as

$$S^{SO(3)}_{\mathrm{CS}}(\kappa) = 2\pi\kappa \int_{M_4} p_1 \tag{35}$$

where $\partial M_4 = M_3$ and p_1 is the first Pontryagin class, generator of $H^4(BSO(3), \mathbb{Z})$.

For a principal $SO(3)$-bundle we have the identity

$$p_1 = w_2 \cup w_2 \mod 2 \tag{36}$$

If the bundle can be lifted to an $SU(2)$-bundle then $w_2 = 0$ which means that

$$\frac{p_1}{2} = \lambda \tag{37}$$

is a well defined integral class. But λ can be identified with the generator of $H^4(BSU(2), \mathbb{Z})$, which is used to define the action of $SU(2)$ CS theory

$$S_{\text{CS}}^{SU(2)}(\kappa) = 2\pi\kappa \int_{M_4} \lambda = 2\pi\kappa \int_{M_4} \frac{p_1}{2} = S_{\text{CS}}^{SO(3)}(\kappa/2) \tag{38}$$

If the $SU(2)$ theory is well defined, then gauging the $\mathbb{Z}_2$ 1-form symmetry leads to an $SO(3)$ with half the level. If the original level is odd the quotient theory in not well defined.

In this case we say that the $SU(2)$ CS theory with **odd level** has an **'t Hooft anomaly** for the 1-form symmetry.

Similarly one can work out the generalization to $SU(N)$ CS theory with $\mathbb{Z}_N$ 1-form symmetry.

7 2-groups

The previous examples can be described through the language of 2-groups. Any (strict) 2-group can be modeled as a crossed module of ordinary groups.

A **crossed module** is defined by a quadruple (H, G, t, α) where H and G are ordinary groups and t, α are homomorphisms

$$\begin{aligned} t &: H \to G \\ \alpha &: G \to \mathrm{Aut}(H) \end{aligned} \tag{39}$$

subject to the structure relations

$$\begin{aligned} \alpha_{t(h)}(h') &= h\, h'\, h^{-1} \qquad && h \in H \\ t(\alpha_g(h)) &= g\, t(h)\, g^{-1} \qquad && g \in G \end{aligned} \tag{40}$$

These structure relations imply that

- $\ker(t)$ is an abelian subgroup of H,
- $\mathrm{Im}(t)$ is normal in G.

We can then define Π_1, Π_2 as

$$\begin{tikzcd} \Pi_2 \arrow[r, "\ker(t)"] & H \arrow[d, "t"] & \\ & G \arrow[r, "\mathrm{coker}(t)"] & \Pi_1 \end{tikzcd} \tag{41}$$

so that the crossed module $(\Pi_2, \Pi_1, 0, \alpha)$ defines a **minimal model** for the 2-group. This minimal model describes the symmetries of the IR theory.

The physical interpretation is the following [8][9][10]:

- G is the 0-form symmetry group
- H is the 1-form symmetry group
- t is the action of the 1-form symmetry on the Wilson line operators
- $\mathrm{Im}(t)$ is the *confined* subgroup of G
- $\Pi_1 = G/\mathrm{Im}(t)$ is the effective IR gauge group
- $\Pi_2 = \ker(t)$ is the group of *unconfined* 't Hooft loops (IR magnetic fluxes)

7.1 Example 1

We now describe the abelian Chern-Simons theory with the 1-form symmetry gauged as a (T)QFT associated to a 2-group.

The crossed module is given by $(\mathbb{Z}_\kappa, U(1), i, 0)$ with $i : \mathbb{Z}_\kappa \to U(1)$ the canonical inclusion.

The minimal model is given by

$$\begin{array}{ccccc} 0 & \xrightarrow{\ker(i)} & \mathbb{Z}_\kappa & & \\ & & \downarrow^{i} & & \\ & & U(1) & \xrightarrow{\mathrm{coker}(i)} & U(1)/\mathbb{Z}_\kappa \end{array} \tag{42}$$

The IR theory is an ordinary $U(1)/\mathbb{Z}_\kappa$ gauge theory with no 1-form symmetry.

7.2 Example 2

The non-abelian $SU(2)$ Chern-Simons theory is described by the crossed module $(\mathbb{Z}_2, SU(2), i, 0)$ with $i : \mathbb{Z}_2 \to SU(2)$ the inclusion of the center.

The minimal model is an ordinary $SO(3)$ gauge theory

$$\begin{array}{ccccc} 0 & \to & \mathbb{Z}_2 & & \\ & & \downarrow^{i} & & \\ & & SU(2) & \xrightarrow{\pi} & SO(3) \end{array} \tag{43}$$

where the 1-form symmetry has been lifted.

A similar story holds for the 2-group $(U(1), U(N), i, 0)$ with minimal model $(0, PSU(N), 0, 0)$.

7.3 Example 3

An example in which Π_2 is non-trivial is that of the 2-group

$$\begin{array}{ccccc} \mathbb{Z}_N & \xrightarrow{i} & U(1) & & \\ & & \downarrow^{\cdot N} & & \\ & & U(1) & \to & 0 \end{array} \tag{44}$$

This is a 2-group gauge theory with a $U(1)$ 2-form connection and a $U(1)$ 1-form connection, such that the Wilson lines have charge N under the 1-form symmetry.

The IR theory is equivalent to a 2-group gauge theory for a $\mathbb{Z}_N$ 1-form symmetry acting on nothing. The gauge group G is completely confined.

The path integral is a sum over $H^2(M_3, \mathbb{Z}_N)$.

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